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A Curious Function of Otto Frisch

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by

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ABSTRACT

If $\{x_n\}$ is a countable set of points everywhere dense on $(0,1)$, and $\{j_n\}$ a set of corresponding positive numbers with $\sum_1^\infty j_n = J < \infty$, then a monotone nondecreasing function $f(x)$ on $[0,1]$, with $f(0) = 0$, $f(1) = J$, which is continuous on $[0,1]$ except at each x_n, where it is right continuous, but left discontinuous with jump j_n, is necessarily a sum of step functions uniquely determined by the x_n, j_n. A curious function, explicitly defined by O. Frisch, is proved to be such a function, with jump $j(p/q) = 1/(2^q - 1)$ at each (reduced) rational point p/q on $(0,1)$, and continuous elsewhere. Moreover, it is rational valued iff x is rational, and has some remarkable number theoretic properties, stemming from its character as a sum of step functions.

I. STRUCTURE OF A MONOTONE FUNCTION

In this part, $\{x_n\}$ denotes a fixed, countable set of points everywhere dense on $(0,1)$, and $\{j_n\}$ a set of corresponding numbers $j_n > 0$, with finite sum $\sum_1^\infty j_n = J$. Such a set of pairs x_n, j_n serves to define a monotone function $s(x)$ as a sum of step functions.

<u>Theorem 1</u>. If $s_n(x)$ is the step function

$$s_n(x) = \begin{cases} 0, & 0 \leq x < x_n \\ j_n, & x_n \leq x \leq 1, \end{cases}$$

then $s(x) = \sum_1^\infty s_n(x)$ is well defined on $[0,1]$, and

(a) $s(x)$ is strictly increasing on $[0,1]$, with $s(0) = 0$, $s(1) = J$,

(b) $s(x)$ is continuous on $[0,1]$ except at the points x_n,

(c) at x_n, $s(x)$ is right continuous, but left discontinuous with jump J_n.

Most of these properties of $s(x)$ are proved in Ref. 1, p. 129, and the rest are easily established.

In fact, $s(x)$ is the <u>only</u> such function, in the sense of the following theorem.

<u>Theorem 2</u>. If $f(x)$ is a monotone nondecreasing function on $[0,1]$, with $f(0) = 0$, $f(1) = J$, which is continuous except at the points x_n, and at x_n is right continuous but left discontinuous with jump j_n, then

$$f(x) = \sum_{x_m \leqslant x} j_m, \quad 0 < x \leqslant 1 \quad . \tag{1}$$

Hence $f(x)$ is uniquely determined by the x_n, j_n, and must be the function $s(x)$ of Theorem 1, with all its properties.

Proof. Since $f(x) - f(0) \geqq \sum_{x_m \leqslant x} j_m$ and $f(1) - f(x) \geqq \sum_{x_m > x} j_m$ (Ref. 2, p. 205), we have $f(1) = f(1) - f(0) \geqq \sum_1^\infty j_n = J$. Since we have stipulated that $f(1) = J$, the equality (1) follows, and the rest is obvious.

<u>Note</u>. The condition $f(1) = J \equiv \sum_1^\infty j_n$ is essential. For, if $s(x)$ is the function of Theorem 1, then $f(x) = s(x) + ax$, $a > 0$, satisfies all the other conditions of Theorem 2. It is curious that, if we take $j_n = \varepsilon/2^n > 0$, then $f(1) = \varepsilon + a$, where the sum ε of the jumps (the <u>only</u> discontinuities) may be arbitrarily small, and the total variation $f(1) - f(0)$ arbitrarily large.

II. A STRANGE FUNCTION OF OTTO FRISCH

In a private communication to N. Metropolis, O. Frisch has defined a very curious function $f(x)$: For x on $[0,1]$, and $n = 0,1,2, \ldots$, let $x_n = nx$, $g_n = [x_n]$,

$$b_n = \begin{cases} 0 \text{ if } g_n = g_{n+1} \\ 1 \text{ if } g_n < g_{n+1} \end{cases},$$

and $f(x) = \sum_1^\infty b_n/2^n$. This function has the properties of the following theorem, some of which were stated (without proof) by Frisch.

Theorem 3. The function $f(x)$ is right continuous at $x = 0$, with $f(0) = 0$, and left continuous at $x = 1$, with $f(1) = 1$; strictly increasing on $[0,1]$; rational at every (reduced) rational point p/q on $(0,1)$, where it is right continuous but left discontinuous with a jump $j(p/q) = 1/(2^q - 1)$, the sum of all jumps being unity; and continuous and irrational at every irrational x on $(0,1)$.

The theorem is in the nature of a summary, the truth of which will become apparent from the following remarks.

1. $f(1) = 1$. For $x = 1$, one has $x_n = n$, $g_n = n$, $b_n = 1$ for $n \geq 0$ and $f(1) = \sum_1^\infty 1/2^n = 1$.

2. For $0 \leq x < 1$, we may write $f(x) = \sum_0^\infty b_n/2^n$. For then $x_0 = 0$, $x_1 = x < 1$, $g_0 = 0$, $g_1 = 0$, and $b_0 = 0$. (It is convenient to include $b_0 = 0$ in the definition of $f(x)$ for $x < 1$, and we do so hereafter.)

3. $f(0) = 0$. For $x = 0$, one has $x_n = 0$, $g_n = 0$, $b_n = 0$ for $n \geq 0$, and $f(0) = \sum_0^\infty 0/2^n = 0$.

4. For any x on $[0,1]$, the sequence $g_0, g_1, g_2, \ldots$, can jump by at most 1. For, $g_n \leq nx < g_n + 1$ implies $g_n \leq nx + x < (g_n + 1) + x \leq g_n + 2$. Thus $g_n \leq (n + 1)x < g_n + 2$ and g_{n+1} is g_n or $g_n + 1$.

5. If $x > 0$, the sequence $b_0, b_1, b_2, \ldots$, cannot terminate in 0. For, $x_n = nx \to \infty$ as $n \to \infty$. Hence $g_n \to \infty$, whereas termination of $\{b_n\}$ in 0 implies all g_n identical after some point.

6. If $x < 1$, the sequence $b_0, b_1, b_2, \ldots$, cannot terminate in 1. If $1 = b_N = b_{N+1} = \ldots$ for some N, we should have $g_{N+k} = g_N + k$ for $k = 1, 2, 3, \ldots$ by (4), and hence $g_N + k \leq (N + k)x < (g_N + k) + 1$, or $g_N/k + 1 \leq (N/k + 1)x < (g_N + 1)/k + 1$. Since N is fixed, and $k = 1, 2, 3, \ldots$, this implies $1 \leq x \leq 1$ and $x = 1$.

7. $f(x)$ is strictly increasing on $[0,1]$. By (5) and (6), it suffices to prove $f(x) < f(x')$ for $0 < x < x' < 1$. Since $x_n = nx \leq nx' = x'_n$ for all $n \geq 0$,

we must have $g_n \leqslant g'_n$ for all n. Moreover, there must be a first N for which $g_N < g'_N$. Otherwise we should have all $g_n = g'_n$ whence $g_n \leqslant nx < g_n + 1$ and $g_n \leqslant nx' < g_n + 1$. Writing the first of these as $-g_n - 1 < -nx \leqslant -g_n$ and adding to the second gives $-1 < n(x' - x) < 1$, a contradiction for $x' > x$ and $n \to \infty$. Thus for x and x', the g_n sequences are of form (cf. (2)) $g_0 = 0$, $g_1 = 0$, $g_2, \ldots, g_{N-1}, g_N, \ldots$; $g_0 = 0$, $g_1 = 0$, $g_2, \ldots, g_{N-1}, g'_N, \ldots$, where $g_N < g'_N$ for an $N \geqslant 2$. Now if $g_{N-1} < g_N$, then $g'_N > g_N > g_{N-1}$; g'_N would be at least 2 more than its immediate predecessor, which is impossible by (4). Hence we must have $g_N = g_{N-1}$, $g'_N = g_{N-1} + 1$, and so $b_{N-1} = 0$, $b'_{N-1} = 1$. It then follows from (5) and (6) that $f(x) < f(x')$.

8. f(x) is left continuous at x = 1. For x = 1 and $n \geqq 1$, we have the defining sequences

x_n :	1	2	3	...	N-1	N	N+1	...
g_n :	1	2	3	...	N-1	N	N+1	...
b_n :	1	1	1	...	1	1	1	... .

For small $\varepsilon > 0$, and N the first integer for which $(N + 1)\,\varepsilon > 1$, the corresponding sequences for $x' = 1 - \varepsilon$ will read

x'_n :	$1-\varepsilon$	$2-2\varepsilon$	$3-3\varepsilon$	...	$(N-1) - (N-1)\varepsilon$	$N-N\varepsilon$	$(N+1) - (N+1)\varepsilon$	...
g'_n :	0	1	2	...	N-2	N-1	N-1	...
b'_n :	1	1	1	...	1	0	... .	

Clearly $f(1 - \varepsilon) \to 1 = f(1)$ as $\varepsilon \to 0$.

9. f(x) is right continuous at x = 0. For x = 0 and $n \geqslant 1$,

x_n :	0	0	0	...	0	0	0	...
g_n :	0	0	0	...	0	0	0	...
b_n :	0	0	0	...	0	0	0	... .

For small $x' = \varepsilon > 0$, and N the first integer for which $(N + 1)\,\varepsilon \geqslant 1$, we have the sequences

x'_n :	ε	2ε	3ε	...	$(N-1)\varepsilon$	$N\varepsilon$	$(N+1)\varepsilon$	...
g'_n :	0	0	0	...	0	0	1	...
b'_n	0	0	0	...	0	1	... ,	

so that $f(\varepsilon) \to 0 = f(0)$ as $\varepsilon \to 0$.

10. If $x = p/q$ is a reduced fraction on (0,1), i.e., $q \geqq 2$, $1 \leqslant p < q$, $(p,q) = 1$, then

$$g_{kq+i} + p = g_{(k+1)q+i} \quad ,$$

for $i = 0, 1, 2, \ldots, q-1$ and all $k \geqslant 0$. For, $g \leqslant (kq + i)\, p/q < g + 1$ implies $(g + p) \leqslant ((k + 1) q + i)\; p/q < (g + p) + 1$.

11. For a reduced fraction $x = p/q$ on $(0,1)$, the sequence $\{b_n\}$, $n \geqslant 0$, is pure periodic, with a period of length q, and $f(x)$ is rational. For, by (10), the g_n sequence is of form $g_0, g_1, \ldots, g_{q-1}; g_0 + p, g_1 + p, \ldots, g_{q-1} + p; g_0 + 2p, \ldots$, where $g_0 = g_1 = 0$. Consequently the b_n sequence $b_0, b_1, \ldots, b_{q-1}; b_0, b_1, \ldots, b_{q-1}; b_0, \ldots$ is periodic of period k, and $f(x)$ is rational.

<u>Ex. 1</u>. For $x = 1/2$, one has the sequences

x_n : 0, 1/2; 1, 3/2; 2, 5/2; 3, ...,
g_n : 0, 0 ; 1, 1 ; 2, 2 ; 3, ...,
b_n : 0, 1 ; 0, 1 ; 0, 1 ; 0, ...,
$f(1/2) = 1/2 + 1/2^3 + 1/2^5 + \ldots = 2/3$.

<u>Ex. 2</u>. For $x = 2/5$,

x_n : 0, 2/5, 4/5, 6/5, 8/5; 10/5, 12/5, 14/5, 16/5, 18/5; 20/5, ...,
g_n : 0, 0, 0, 1, 1; 2, 2, 2, 3, 3; 4, ...,
b_n : 0, 0, 1, 0, 1; 0, 0, 1, 0, 1; 0, ...,
$f(2/5) = (1/2^2 + 1/2^4)\,(1 + 1/2^5 + 1/2^{10} + \ldots) = 10/31$.

12. For a reduced fraction $x = p/q$ on $(0,1)$, one has for all $k \geqslant 1$,

$x_{kq-1} = (kq - 1)\; p/q = kp - p/q$, $g_{kq-1} = kp - 1$,
$x_{kq} = (kq)\; p/q = kp$, $g_{kq} = kp$,
$x_{kq+1} = (kq + 1)\; p/q = kp + p/q$, $g_{kq+1} = kp$.

Hence for $x = p/q$, the b_n sequence is always of form $0, b_1, \ldots, b_{q-2}, 1; 0, b_1, b_2, \ldots, b_{q-2}, 1; 0, \ldots$.

13. For a reduced fraction $x = p/q$ on $(0,1)$, we have

$f(p/q) = (B + 1/2^{q-1})/(1 - 1/2^q)$, where $B = \sum_1^{q-2} b_n/2^n$ ($B = 0$ for $x = 1/2$). This is clear from (12).

14. If $x = p/q$ is a reduced fraction on $(0,1)$, and $\varepsilon > 0$ is very small, then for $x' = p/q - \varepsilon$, the b'_n sequence will begin with $0, b_1, b_2, \ldots, b_{q-2}, 0; 1, b_1, b_2, \ldots, b_{q-2}, 0; 1, \ldots$, the only change in the initial g_n being from the g - triads $kp - 1, kp, kp$ to the g' triads $kp - 1, kp - 1, kp$, as may be inferred from (12). For $x_n = n(p/q)$ is an integer iff n is a multiple of q.

15. It is clear that the smaller $\varepsilon > 0$ is taken in (14), the longer this b_n - pattern will persist. Hence

$$f((p/q)^-) = \lim_{\varepsilon \to 0} f(p/q - \varepsilon) = (B + 1/2^q)/(1 - 1/2^q) \quad .$$

16. f(x) is left discontinuous at every reduced fraction $x = p/q$ on (0,1), with a left jump

$$j(p/q) = f(p/q) - f((p/q)^{-}) = 1/(2^q - 1) \quad .$$

This follows at once from (13) and (15).

17. f(x) is right continuous at every reduced fraction $x = p/q$ on (0,1). For small $\varepsilon > 0$, the b'_n - pattern for $x' = p/q + \varepsilon$ will begin just as does the b_n - pattern for $x = p/q$, and the smaller $\varepsilon > 0$, the longer this pattern will persist.

18. The sum of all the jumps $j(p/q)$ is unity. Since for $q \geqslant 2$, there are $\phi(q)$ reduced fractions p/q on (0,1), at each of which there is the same jump $j(p/q) = 1/(2^q - 1)$, we must have

$$J \equiv \sum_{0<p/q<1} j(p/q) = \sum_{2}^{\infty} \phi(q)/(2^q - 1) = 1 \quad .$$

The value 1 of the series may be inferred by setting $y = 1/2$ in the Liouville identity (Ref. 3, p. 240)

$$\sum_{1}^{\infty} \phi(q)\, y^q/(1 - y^q) \equiv y/(1 - y)^2, \; |y| < 1 \quad .$$

(The latter is an easy consequence of the well known property $\sum_{d|m} \phi(d) = m$ of the Euler ϕ-function.)

19. f(x) is continuous at every irrational $x = \theta$ on (0,1). For small $\varepsilon > 0$, the b'_n-patterns for both $x' = \theta + \varepsilon$ and for $x' = \theta - \varepsilon$ will begin as does the b_n-pattern for $x = \theta$, the latter because $n\theta$ is never an integer, and the smaller $\varepsilon > 0$ is taken, the further this pattern will continue. Thus f(x) is right and left continuous at $x = \theta$.

20. f(x) is irrational at every irrational $x = \theta$ on (0,1). Suppose on the contrary that for some irrational $x = \theta$, the b_n sequence were terminally periodic of some period length q. Since by (6) it cannot terminate in 1, we should have an N such that for all $k \geqslant 0$,

$$b_{N+kq} = 0,$$

and hence $g_{N+kq} = g_{N+kq+1}$.

Now write $N\theta = g + \delta$, $(N + 1)\theta = g + \varepsilon$,

where $g = g_N$ is an integer, and $0 < \delta < \varepsilon < 1$.

Also let $q\theta = c + \eta$,

where $c = [q\theta]$, $0 < \eta < 1$, and η is irrational.

Then $(N + kq)\theta = g + \delta + kc + k\eta$,

and $(N + 1 + kq)\theta = g + \varepsilon + kc + k\eta$

must have the same integral part, for all $k \geqslant 0$.

Now η is irrational, and the numbers $k\eta$, $k \geqslant 0$ are uniformly distributed mod 1 (Ref. 4, p. 72). Hence there exists a K such that

$$K\eta = h + r \quad ,$$

where h is an integer, and

$$0 < 1 - \varepsilon < r < 1 - \delta < 1 \quad .$$

Hence $r + \delta < 1$ and $r + \varepsilon > 1$.

Then $(N + Kq)\theta = g + \delta + Kc + h + r = (g + Kc + h) + (r + \delta)$, and $(N + 1 + Kq)\theta = g + \varepsilon + Kc + h + r = (g + Kc + h) + (r + \varepsilon)$. But obviously these two numbers do <u>not</u> have the same integral part, and this is a contradiction.

III. NUMBER THEORY AND THE FRISCH FUNCTION

Since the Frisch function has the properties of Theorem 2, we may infer at once the following theorem.

<u>Theorem 4</u>. The Frisch function f(x) is the sum s(x) of the step functions in Theorem 1, where $\{x_n\}$ is the set of all reduced fractions p/q on (0,1), and $j(p/q) = 1/(2^q - 1)$.

Thus f(x) has all the properties of s(x), some of which are easily obtained from the original definition in Part II, as we have seen, but others are by no means obvious.

It is interesting that the value of f(1/2) may be found both from the Frisch definition of f(x), and from its realization s(x). Thus f(1/2) = 2/3, as we saw in Part II (11), whereas

$$s(1/2) = \sum_{\frac{p}{q} \leq \frac{1}{2}} j\,(p/q) = j\,(1/2) + \sum_{3}^{\infty} \frac{(1/2)\,\phi(q)}{2^q - 1}$$

$$= \frac{1}{3} + \frac{1}{2}\left(\sum_{2}^{\infty} \frac{\phi(q)}{2^q - 1} - \frac{\phi(2)}{2^2 - 1}\right) = \frac{1}{3} + \frac{1}{2}\left(1 - \frac{1}{3}\right) = \frac{2}{3} \quad .$$

But $x = 1/2$ is the only fraction on $(0,1)$ for which $s(x)$ is easily evaluated. It seems remarkable therefore that the readily evaluated $f(p/q)$ provides a means of summing the series $s(p/q)$.

Theorem 5. If a/b is a rational number on $(0,1]$, and $N(q, a/b)$ denotes the number of integers p prime to q for which $1 \leq p \leq (a/b) \cdot q$, then

$$\sum_{q \geq b/a} \frac{N(q,a/b)}{2^q - 1} = f(a/b) \quad ,$$

where $f(a/b)$ is evaluated in Part II (13).

Note that $N(q,1) = \phi(q)$ for $q \geq 2$, and $N(q,1/2) = \frac{1}{2}\phi(q)$ for $q \geq 3$, but for other values of a/b, $N(q,a/b)$ seems to be a new arithmetic function with interesting properties. As a final example (cf. Part II (11), we note that

$$10/31 = f(2/5) = s(2/5) = \sum_{q=3}^{\infty} \frac{N(q,2/5)}{2^q - 1}$$

$$= 1/7 + 1/15 + 2/31 + 1/63 + 2/127 + 2/255$$

$$+ 2/511 + 2/1023 + 4/2047 + 1/4095 + 5/8191 + \ldots \quad .$$

Note. The Frisch function seems a fitting companion to a similarly constructed function of van der Waerden, which is everywhere continuous, but nowhere differentiable (Ref. 5, p. 353).

Note added in proof: A short discussion of the function by Frisch will appear in Ref. 6.

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